$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = 0$$
$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

<u>Definition</u>: (linearly Dependent Row (Column) vectors): Row vectors $R_1, R_2, \ldots, R_m$ are said to be linearly dependent (L.D.) if there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ not all zero such that

$$\alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m = 0$$

Similarly, column vectors $C_1, C_2, \ldots, C_n$ are L.D. if $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all zero such that

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n = 0$$

<u>Definition</u>: (linear combination of Row (Column) vectors):
A row vector R is called a linear combination of row vectors $R_1, R_2, \ldots, R_m$ if $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ such that

$$R = \alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m.$$

Similarly, a column vector C is called a linear combination of column vectors $C_1, C_2, \ldots, C_n$ if $\exists$ scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$C = \alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_n C_n.$$

Thm. Vectors $(a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{n1}, a_{n2}, \ldots, a_{nn})$ are linearly dependent iff

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Note: This Theorem is valid only for square matrix.

<u>Proof</u>: Consider the linear combination

$$\alpha_1 (a_{11}, a_{12}, \ldots, a_{1n}) + \alpha_2 (a_{21}, a_{22}, \ldots, a_{2n}) + \cdots + \alpha_n (a_{n1}, a_{n2}, \ldots, a_{nn}) = (0, 0, \ldots, 0) \quad \text{--- (1)}$$

$$\Rightarrow (\alpha_1 a_{11} + \alpha_2 a_{21} + \cdots + \alpha_n a_{n1}, \ \alpha_1 a_{12} + \alpha_2 a_{22} + \cdots + \alpha_n a_{n2}, \ \ldots, \ \alpha_1 a_{1n} + \alpha_2 a_{2n} + \cdots + \alpha_n a_{nn}) = (0, 0, \ldots, 0)$$

$$\Rightarrow \alpha_1 a_{11} + \alpha_2 a_{21} + \cdots + \alpha_n a_{n1} = 0$$

$$\alpha_1 a_{12} + \alpha_2 a_{22} + \cdots + \alpha_n a_{n2} = 0$$

$$\vdots$$

$$\alpha_1 a_{1n} + \alpha_2 a_{2n} + \cdots + \alpha_n a_{nn} = 0$$

which are n linear homogeneous equations in n variables $\alpha_1, \alpha_2, \dots \alpha_n$.

These equations have a non-trivial solution iff determinant of coefficients is zero.

i.e. iff
$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

i.e.
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

iff
$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{vmatrix} = 0$$

i.e.
$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

has non trivial solution iff

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} = 0$$

$\Rightarrow$ In equation ①, the linear Combination is true for scalars $\alpha_1, \alpha_2, \cdots \alpha_n \in F$ not all zero iff

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

i.e. the vectors $(a_{11}, a_{12}, \cdots a_{1n})$, $(a_{21}, a_{22}, \cdots a_{2n})$, $\cdots$, $(a_{n1}, a_{n2}, \cdots a_{nn})$ are L.D.

iff $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

Similarly, Column vectors of a square matrix A are linearly dependent iff $|A| = 0$.